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PPM based UWB System Throughput Optimisation

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Abstract— **The system throughput performance of an ultra-wideband (UWB) communication system applying binary pulse position modulation (PPM) is investigated. The optimum impulse shift is analytically derived for the AWGN channel. The information throughput of an UWB system in AWGN channel has been optimized by adjusting the time overlap of the pulse shape used according to the actual signal-to-noise ratio (SNR).**

Keywords— **Ultra-Wideband, Pulse Position Modulation, Performance Analysis**

I. Introduction

The Ultra-wideband technology was first patented for communications in 1961 by Hoeppner [1]. Since then many attempts have been reported for building up short to medium range data transmission systems applying UWB at the physical layer (PHY) [2], [3]. The main advantages of UWB systems are the potential for very high data rates at low power operation and the possibility for real time subdecimeter positioning [4], [5]. Recent developments shift the technology from military to civilian applications [6], [7], [8]. Regulation bodies in the US and Europe are on the way to release rules for UWB systems operation [9], [10], [11].

An UWB system applying time spreading of bits into chips is considered, where each chip is represented by one impulse. Furthermore is soft integration of the chip correlation results assumed before bit decision. Blanking is not necessary due to the additive white Gaussian noise (AWGN) channel and a single user case assumption [12]. A loss in bit error ratio performance due to a increased sensitivity against AWGN caused by a impulse overlap can be compensated by increasing the number of chips per bit. Such a time overlap between possibly transmitted symbols is introduced in order to shorten transmit symbol time $\mathrm{T_S}$. Therefore it must be analyzed which loss in bit error ratio results from that and how much more bit energy is needed for compensation. The aim of this work is to get the optimum value of symbol duration $\mathrm{T_S}$ for a maximal average channel information throughput I as a function of a certain impuls shape $\mathrm{i}(\mathrm{t})$ and the signal-to-noise ratio SNR.

PPM, also called disjoint PPM (DJPPM) [13] is one of the basic modulation schemes with information coded into time of appearance of a certain impulse in one out of M possible disjointed time slots. Maximizing the channel information throughput for DJPPM is achieved by shortening the impulse duration τ_w and increasing the symbol alphabet M [14]. Therefore the transmit symbol time for DJPPM can in general be given by (1).

$$\mathrm{T_{S,DJPPM}} = \tau_\mathrm{w} \cdot \mathrm{M} \tag{1}$$

An other approach is useful in practical cases, where the impulse width is fixed and power constraints exist. This is overlapping PPM (OPPM) [15], that splits the transmit symbol time into M overlapping time slots. The value of overlap in time is considered to be τ_o. Thus another useful variable for the time shift between impulses that follow one after the other may be introduced as well. This is $\mathrm{t_s} = \tau_\mathrm{w} - \tau_\mathrm{o}$. Summarizing the transmit symbol time for OPPM can be given by (2).

$$\mathrm{T_{S,OPPM}} = \mathrm{t_s} \cdot (\mathrm{M} - 1) + \tau_\mathrm{w} \tag{2}$$

Applying OPPM the signals at the input of the decision device in the optimal correlator receiver become correlated and the decisions will be more channel noise sensitive keeping the the impulse energy constant. Hence the bandwidth efficiency can be increased at expense of a rised sensitivity against AWGN however. As one can expect, there should exist an optimal overlap depending on the signal-to-noise ratio and the impulse shape used. Starting with no overlap the transmit symbol time $\mathrm{T_S}$ is quite large while the AWGN sensitivity is minimal. The opposite extreme case is the almost total overlap yielding minimal $\mathrm{T_S}$ but maximal sensitivity against AWGN. In between an optimal overlap giving the maximal average channel information throughput should exist. Deriving this optimal overlap analytically for the typical UWB received impulse shape [16] and the special case of binary OPPM is the intention of the following analysis subsection.

II. UWB Information Theory

A. Basic Approach

In [17] the error ratio performance of PPM, MPPM and OPPM in the indoor wireless optical channel has been investigated for a rectangular pulse shape. The error performance of M-ary modulation was identified assuming high SNR environment allowing simplified analysis based on the minimum distance criterium. The general spirit of the intense background light channel model used in [17] and may also be applied to the UWB evaluation assuming AWGN as the only distortion of the received signal. But a high SNR assumption is inapplicable as well as the impulse shape $\mathrm{i}(\mathrm{t})$ is far more complex in the UWB case [16]. A variable τ_n may represent a time normalization to keep $\mathrm{i}(\mathrm{t})$ independent from a specific impuls duration.

$$\mathrm{i}\left(\frac{\mathrm{t}}{\tau_\mathrm{n}}\right) = \left(1 - 4\pi\left(\frac{t}{\tau_\mathrm{n}}\right)^2\right)\exp\left(-2\pi\left(\frac{t}{\tau_\mathrm{n}}\right)^2\right) \tag{3}$$

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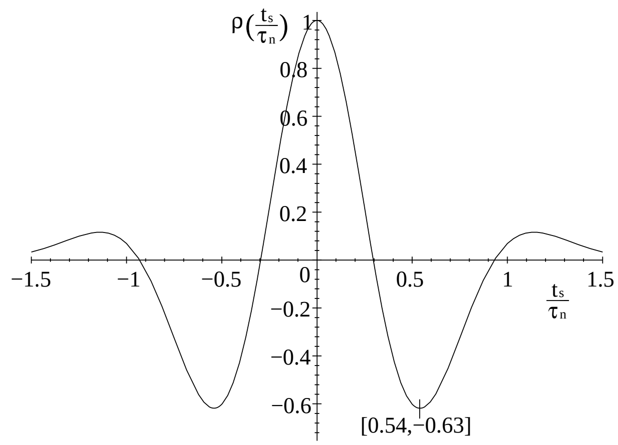

Fig. 1. correlation function $\rho\left(\frac{\mathrm{t_s}}{\tau_\mathrm{n}}\right)$ versus time shift $\frac{\mathrm{t_s}}{\tau_\mathrm{n}}$

If a signal described in (3) is partly overlapped with its replica like in the OPPM case the correlation between two of those signals versus time shift is shown in figure 1. The analytical expression describes (4) [16].

$$\rho\left(\frac{\mathrm{t_s}}{\tau_\mathrm{n}}\right) = \left(1 - 4\pi\left(\frac{\mathrm{t_s}}{\tau_\mathrm{n}}\right)^2 + \frac{4\pi^2}{3}\left(\frac{\mathrm{t_s}}{\tau_\mathrm{n}}\right)^4\right)\exp\left(-\pi\left(\frac{\mathrm{t_s}}{\tau_\mathrm{n}}\right)^2\right) \tag{4}$$

Considering two correlators as shown in Fig.2 the input into the decision device will be the difference of both correlation paths.

Since these values include the information as well as a noise part the channel variance parameter ${\sigma_C}^2$ has to be transformed from the input of the correlator to its output for a later analysis. Doing so the variance at the input of the decision device ${\sigma_D}^2$ becomes a function of the correlation $\rho\left(\mathrm{t_s}/\tau_\mathrm{n}\right)$ and the impuls energy $\mathrm{E_i}$. For the full evaluation see appendix A.

Fig. 3 shows that the correlator now converts binary data into analog signal values at certain times in front of the noise source of ${\sigma_D}^2$. So the discrete-time AWGN memoryless channel can be used further on. The average mutual information between the discrete input $X = \{x_0, \ldots, x_\mathrm{M}\}$ and continues output $Y = \{-\infty, \infty\}$ is given [18, eq.7-1-18]

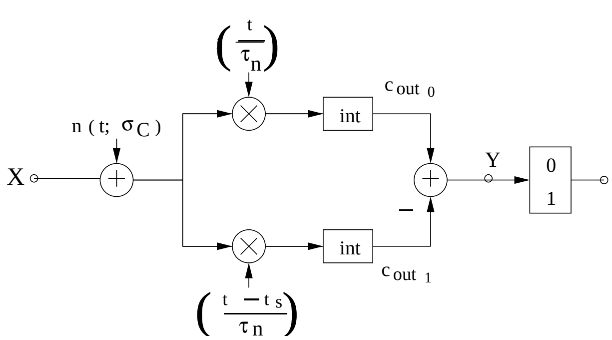

Fig. 2. transmission path including two correlators

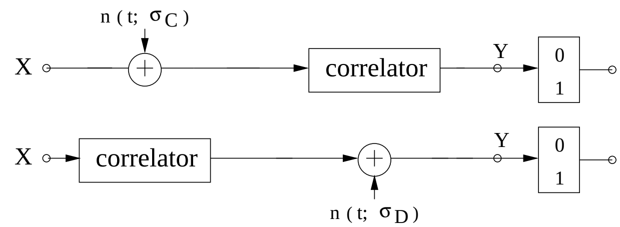

Fig. 3. transmission path before and after the noise tranformation

by the capacity of this channel in bits/channel use as

$$\mathrm{C}\left[\frac{\mathrm{bit}}{\mathrm{sym}}\right] = \max_{\mathrm{P(x_i)}} \sum_{i=0}^{\mathrm{M}-1} \int_{-\infty}^{\infty} \mathrm{p(y\,|\,x_i)P(x_i)}\log_2\frac{\mathrm{p(y\,|\,x_i)}}{p(y)}\,dy\,. \tag{5}$$

Within this equation $\mathrm{P(x_i)}$ represents the probability of occurrence of the channel input symbol $\mathrm{x_i}$, which is defined by the information source. Furthermore $\mathrm{p(y\,|\,x_i)}$ is the conditional probability density function defined by the discrete input continuous output transmission channel.

B. Analysis

For the channel model used with equal energy input symbols the maximum of the right hand side of (5) is achieved, if the input symbols are equally distributed. Thus for the binary case $\mathrm{P(x_i)} = 1/2$.

The meaning of $\mathrm{x_i}$ in (5) is, that it represents the expected input at the decision device in case there is no distortion of the transmitted signal. In the binary case with equal energy signals considered here this input is the difference of the two correlator outputs.

$$\mathrm{E_D} = c_\mathrm{out}(x_0) - c_\mathrm{out}(x_1) \tag{6}$$

This output depends on the correlation between possibly transmitted symbols which itself is a function of the transmitted pulse shape $\mathrm{i}\left(\mathrm{t}/\tau_\mathrm{n}\right)$ and the normalized time shift $\frac{\mathrm{t_s}}{\tau_\mathrm{n}}$. A new variable $\mathrm{E_D}$ is introduced describing the expected output of the correlator in the undisturbed case. Then (5) becomes

$$\begin{aligned}\mathrm{C(t_s)} \quad = \quad & \frac{1}{2}\int_{-\infty}^{\infty} p(y\,|\,\mathrm{E_D})\log_2\frac{p(y\,|\,\mathrm{E_D})}{p(y)}\,dy + \\ & \frac{1}{2}\int_{-\infty}^{\infty} p(y\,|{-\mathrm{E_D}})\log_2\frac{p(y\,|{-\mathrm{E_D}})}{p(y)}\,dy\,. \end{aligned} \tag{7}$$

The capacity is a function of the SNR, of the time shift $\mathrm{t_s}$ and of the pulse shape. With increasing SNR the capacity increases as well. This holds for any time shift and any impulse shape. The capacity is expressed in bits per channel input symbol. In case of binary input it can become 1 at maximum. To get a criterium for the system efficiency this capacity needs to be multiplied by the number of input symbols per time unit. Then the normalized capacity does incorporate the gain in time by shortening the symbol time compared to the orthogonal transmission case, where the symbol time is twice the impulse width (like in DJPPM). As the time shift t_s increases the number of input symbols per fixed time unit increases as well.

The probability density function (pdf) at the decision device is the sum of the two Gaussian pdfs $p(y\,|\,E_D)$ and $p(y\,|\,-E_D)$ with variance ${\sigma_D}^2$ around the expected values at the decision device, E_D and $-E_D$, respectively. By introducing $\Phi(x_G)$ as a Gauss function with the requested properties,

$$\Phi(x_G) = \frac{1}{2\sqrt{2\pi}\sigma_D} e^{-\frac{{x_G}^2}{2{\sigma_D}^2}} \tag{8}$$

the following actual distributions can be used with (7):

$$\begin{aligned} p(y\,|\,E_D) &= \Phi(x + E_D) & (9)\\ p(y\,|\,{-E_D}) &= \Phi(x - E_D) & (10)\\ p(y) &= \frac{1}{2}\left(\Phi(x+E_D) + \Phi(x+E_D)\right) & (11) \end{aligned}$$

$H(Y)$ can now be written as

$$\begin{aligned} H(Y) = \\ &-\frac{1}{2}\int_{-\infty}^{\infty} \Phi(x+E_D)\cdot \log_2\left(\frac{\Phi(x+E_D)+\Phi(x-E_D)}{2}\right)dx \\ &-\frac{1}{2}\int_{-\infty}^{\infty} \Phi(x-E_D)\cdot \log_2\left(\frac{\Phi(x+E_D)+\Phi(x-E_D)}{2}\right)dx \end{aligned} \tag{12}$$

The remaining terms of (7) represent the entropy that was added by the channel an can be written as

$$\begin{aligned} H(Y|X) = \\ &\frac{1}{2}\int_{-\infty}^{\infty} \Phi(x+E_D)\cdot \log_2\Big(\Phi(x+E_D)\Big)dx \\ &+\frac{1}{2}\int_{-\infty}^{\infty} \Phi(x-E_D)\cdot \log_2\Big(\Phi(x-E_D)\Big)dx \end{aligned} \tag{13}$$

The capacity can thus be expressed as

$$C(t_s) = 1 - \log_2\sqrt{e} - \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\cdot g(t)\; dt \tag{14}$$

where

$$\begin{aligned} g(t) = \log_2\Bigg(& e^{-2t^2} + e^{-\left(t^2+\left(t-\frac{2E_D}{\sqrt{2}\sigma_D}\right)^2\right)} \\ & + e^{-\left(t^2+\left(t+\frac{2E_D}{\sqrt{2}\sigma_D}\right)^2\right)} + e^{-\left(2t^2+2\left(t+\frac{2E_D}{\sqrt{2}\sigma_D}\right)^2\right)}\Bigg) \end{aligned} \tag{15}$$

It is shown in Appendix A that the noise at the decision device is a gaussian process with the variance ${\sigma_D}^2 = 2E_i \cdot {\sigma_C}^2 \cdot (1-\rho)$. The part of the output that depends on the

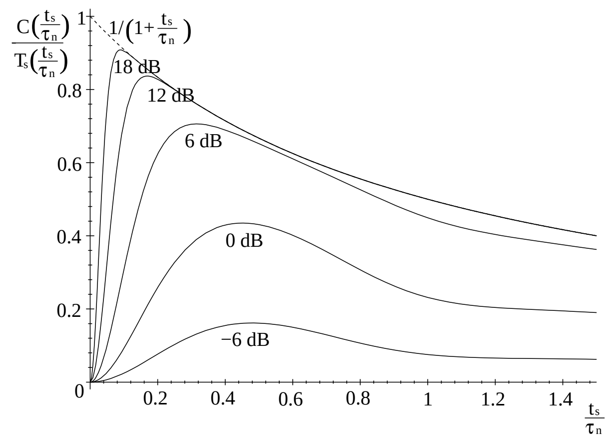

Fig. 4. C $\left[\frac{\text{bit}}{\text{s}}\right]$ versus time shift $\frac{t_s}{\tau_n}$ for various channel SNR

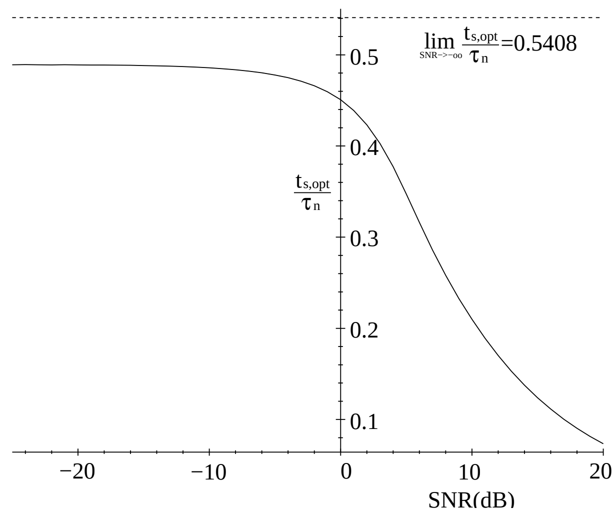

Fig. 5. optimal time shift $\frac{t_{s,opt}}{\tau_n}$ versus channel SNR

transmitted symbol is shown to be $E_D = (1-\rho)E_i$. As $g(t)$ depends only on the relation of these terms, and not on either one of them seperately, $C(t_s)$ is a function of the signal–to–noise ratio.

In Fig. 4 the mutual information normalized to a fixed time unit versus the time shift between two possible transmit symbols is presented. This is calculated for the impulse shape described by (3) for different SNR. As shown in Fig. 4 there exist a maximum in mutual information for a certain time shift, which depends on the SNR. The higher the SNR, the smaller is the optimum time shift.

In Fig. 5 the optimum time shift versus the SNR is shown for the impulse shape described by (3). It can be seen, that with decreasing SNR the value asymptotically converges to a limit, which is 0.5408. This is due to the nature of the received pulse shape yielding a minimal correlation at this time shift (see Fig. 1).

III. Conclusions

The optimal impulse shift position in case of binary overlapping PPM (2OPPM) has been analytically derived for the AWGN channel for different impulse shapes. It is shown that the channel capacity of an UWB system in AWGN channel can be optimized by adjusting the time overlap of the pulse shape used according to the actual SNR. Especially in the case of very high data rate transmission (like video in home environment) where higher SNR and small spreading is assumed this dependency should be considered in system design in order to optimally use the channel and thus maximize system information throughput.

IV. Acknowledgement

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References

[1] C. H. Hoeppner, "Pulse communication system," U.S. Patent Nr. 2999128, September 1961, filed 14. November 1945.

[2] Robert J. Fontana, "Recent applications of ultra wideband radar and communications systems," to be published in Ultra-Wideband, Short-pulse Electromagnetics, Kluwer Academic/Plenum Publishers.

[3] John McCorkle, "A tutorial on ultrawideband technology," available at http://grouper.ieee.org/groups/802/15/pub/Tutorials.html, March 6th 2000, IEEE 802.15 tutorial.

[4] Paul Withington, "Uwb applications, demonstration and regulatory update," workshop presentation, CEPT workshop on regulatory issues regarding introduction of UWB in Europe, March 2001, Mainz, Germany.

[5] Robert Fleming and Cherie Kushner, "Integrated ultra-wideband localizers," in *UWB Conference*, Washington D.C., September 28 1999, available at http://www.aetherwire.com/Aether_Wire/Integrated_Ultra-Wideband_Localizers.pdf.

[6] Timothy Jason Shepard, *Decentralized Channel Management in Scalable Multihop Spread-Spectrum Packet Radio Networks*, Ph.D. thesis, Massachusetts Institute of Technology, 1995.

[7] Inc. Fantasma Networks, "High performance wireless home networks: An ultra-wideband solution," http://www.fantasmanetworks.com/pdfs/Chipset2001, White Paper.

[8] H. Luediger, S. Zeisberg, M.G. di Benedetto, and N. Blefari Melazzi, "Outline of an open radio access network," in *IEEE Proc. of European Wireless 2000*, Dresden, Germany, September 2000, IEEE, pp. 77–83.

[9] "Notice of proposed rulemaking," 2000, FCC 00-163.

[10] Comission of the European Community, *1st European Ultra Wideband Workshop*, Brussels, December 2000, http://www.cordis.lu/ist/ka4/mobile/pubs-reports.htm#1uwbworkshop.

[11] Reg TP, German Regulatory Authority for Telecommunications and Post, *CEPT workshop on regulatory issues regarding introduction of UWB in Europe*, Meinz, March 2001.

[12] S.S. Kolenchery, J.K. Townsend, J.A. Freebersyser, and G Bilbro, "Performance of local power control in peer-to-peer impulse radio networks with bursty traffic," in *IEEE Global Telecommunications Conference*. IEEE, 1997, pp. 910 –916, GLOBECOM'97.

[13] Israel Bar-David and Gideon Kaplan, "Information rates of photon-limited overlapping pulse position modulation channels," *IEEE Transactions on Information Theory*, vol. IT-30, no. 3, pp. 455–463, May 1984.

[14] J. R. Pierce, "Optical channels: practical limits with photon counting," *IEEE Transactions on Communications*, vol. COM-26, no. 12, pp. 1819–1821, December 1978.

[15] G. M. Lee and G. W. Schroeder, "Optical ppm with multiple positions per pulsewidth," *IEEE Transactions on Communications*, vol. COM-25, pp. 360–364, March 1977.

[16] Fernando Ramirez-Mireles and Robert A. Scholtz, "System performance analysis of impulse radio modulation," in *IEEE Radio and Wireless Conference*. IEEE, August 1998, pp. 67–70, RAWCON'98.

[17] Hyuncheon Park and John R. Barry, "Modulation analysis for wireless infrared communications," in *IEEE International Conference on Communications*, Seattle, June 1995, IEEE, pp. 1182–1186, ICC'95.

[18] John G. Proakis, *Digital Communications*, Electrical Engineering. McGraw-Hill International Editions, 3. edition, 1995.

[19] Horst Stoecker, *Taschenbuch mathematischer Formeln und moderner Verfahren*, Verlag Harri Deutsch, 3. edition, 1995.

[20] A. M. Mood, *Introduction to the Theory of Statistics*, probability and statistics. McGraw-Hill, 3. edition, 1974.

Appendices

I. Variance Transformation

In order to determine which symbol was probably sent based on the received signal, it is common for PPM to correlate the received signal $r(t)$ with templates. These templates should have the same shape as the pulse used by the sender, $\mathrm{i}\,(\mathrm{t})$. The correlator is applied for the possible positions of the pulse. The most likely symbol should yield the greatest correlation coefficient, for the binary case this results in the decision described in (16)

$$\int_0^{\mathrm{T_S}} r(t)\,\mathrm{i}\,(\mathrm{t})\,dt \underset{x_1}{\overset{x_0}{\gtrless}} \int_0^{\mathrm{T_S}} r(t)\,\mathrm{i}\,(\mathrm{t}-\mathrm{t_s})\,dt \tag{16}$$

$r(t)$ consists of the sent signal $tx(t)$ and the channel noise $n(t)$, which is assumed to be gaussian.

$$r(t) = tx(t) + n(t) \tag{17}$$

Insertion of (16) into (17) yields

$$\underbrace{\int_0^{\mathrm{T_S}} tx(t)\mathrm{i}\,(\mathrm{t})\,dt}_{\text{term } \mathrm{a_1}} + \underbrace{\int_0^{\mathrm{T_S}} n(t)\mathrm{i}\,(\mathrm{t})dt}_{\text{term } \mathrm{b_1}} \underset{x_1}{\overset{x_0}{\gtrless}} \underbrace{\int_0^{\mathrm{T_S}} tx(t)\mathrm{i}\,(\mathrm{t}-\mathrm{t_s})\,dt}_{\text{term } \mathrm{a_2}} + \underbrace{\int_0^{\mathrm{T_S}} n(t)\mathrm{i}\,(\mathrm{t}-\mathrm{t_s})\,dt}_{\text{term } \mathrm{b_2}} \tag{18}$$

For the two terms (a) containing the signal, it is obvious that one of the integrals should represent the energy of the pulse $\mathrm{E_i}$, as either $\mathrm{i}\,(\mathrm{t})$ or $\mathrm{i}\,(\mathrm{t}-\mathrm{t_s})$ represent the signal itself. If the signals in their possible positions $\mathrm{i}\,(\mathrm{t})$ and $\mathrm{i}\,(\mathrm{t}-\mathrm{t_s})$ are correlated by ρ, the other term (b) should return the correlation coefficient ρ of these pulses in the respective position times the energy of the pulse. The value $\mathrm{E_D}$ that feeds the decision device will thus be decreased by $\mathrm{E_i}\cdot\rho$

$$\mathrm{E_D} = (1-\rho)\mathrm{E_i} \tag{19}$$

If the possible pulse positions are correlated, so will be the noise terms. This information will be lost once the integrals over the noise terms are solved in (18). It is, however, possible to calculate the variance of the overall correlated noise which disturbs the signal. The overall noise process can be described as

$$n_D(t) = n(t)\mathrm{i}\,(\mathrm{t}) + \big(- n(t)\mathrm{i}\,(\mathrm{t} - \mathrm{t_s})\big) \tag{20}$$

As shown in appendix B, the sum (difference) of two gaussian distributed variables with zero mean and variance ${\sigma_C}^2$ can be expressed as one zero-mean gaussian process with variance

$$\sigma^2_{sum/difference} = 2\sigma^2(1 \pm \rho) \tag{21}$$

Assuming the variance of the noise process $n(t)$ is σ^2, the variance of each of the two terms in (20) can be shown to be

$$\begin{aligned} \sigma_b^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(t)\mathrm{i}\,(\mathrm{t})n(\tau)\mathrm{i}\,(\tau)\,dt\,d\tau \\ &= {\sigma_C}^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t-\tau)\mathrm{i}\,(\mathrm{t})\mathrm{i}\,(\tau)\,dt\,d\tau \\ &= {\sigma_C}^2 \int_{-\infty}^{\infty} \mathrm{i}\,(\mathrm{t})\mathrm{i}\,(\tau)\,dt = \mathrm{E_i}{\sigma_C}^2 \end{aligned} \tag{22}$$

Their correlation is

$$\begin{aligned} \rho_{b_1 b_2} &= \frac{1}{\sigma_b^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} n(t)\mathrm{i}\,(\mathrm{t})n(t+\tau)\big(-\mathrm{i}\,(\mathrm{t}-\mathrm{t_s}+\tau)\big)\big)\,dt\,d\tau \\ &= \frac{{\sigma_C}^2}{\sigma_b^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t-\tau)\mathrm{i}\,(\mathrm{t})(-\mathrm{i}\,(\mathrm{t}-\mathrm{t_s}+\tau))\,dt\,d\tau \\ &= -\frac{1}{\mathrm{E_i}}\int_{-\infty}^{\infty} \mathrm{i}\,(\mathrm{t})\mathrm{i}\,(\mathrm{t}-\tau)\,dt = -\rho \end{aligned} \tag{23}$$

Thus, the noise at the decision device is a gaussian process with zero mean and variance

$${\sigma_D}^2 = 2\mathrm{E_i}\cdot{\sigma_C}^2\cdot(1-\rho) \tag{24}$$

and therefore the bit error probability for the binary correlated transmission becomes

$$\begin{aligned} \mathrm{P}_{Error} &= \mathrm{Q}\left[\frac{\mathrm{E_i}(1-\rho)}{\sqrt{2\mathrm{E_i}\cdot{\sigma_C}^2(1-\rho)}}\right] \\ &= \mathrm{Q}\left[\frac{\sqrt{\mathrm{E_i}}}{\sigma_C}\sqrt{\frac{1-\rho}{2}}\right] \end{aligned} \tag{25}$$

II. Adding Gaussian distributed variables

Let $f_x(x)$ and $f_y(y)$ be two Gaussian distributed variables with expectation $\mathrm{E}[X] = \mathrm{E}[Y] = 0$ and variance $\mathrm{E}[X^2] = \mathrm{E}[Y^2] = \sigma^2$. Further let them be correlated by $\frac{\mathrm{E}[X\cdot Y]}{\sigma^2} = \rho$. According to [19], page 748, their pdf is

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\exp\left[-\frac{x^2-2\rho xy+y^2}{2\sigma^2(1-\rho^2)}\right] \tag{26}$$

[20], page 185, gives an equation to calculate the pdf of $Z = X + Y$:

$$f_z(z) = \int_{-\infty}^{\infty} f_{xy}(x, z-x)dx \tag{27}$$

By combining (26) and (27) it can be found that

$$f_{xy}(x,y) = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2\sqrt{1-\rho^2}}\exp\left[-\frac{x^2-2\rho x(x-z)+(x-z)^2}{2\sigma^2(1-\rho^2)}\right] \tag{28}$$

The integral (28) was solved and some simplifications were carried out to find that

$$f_{xy}(x,y) = \frac{1}{\sqrt{2\pi}\sigma\cdot\sqrt{2(1+\rho)}}\cdot\exp\left[-\frac{z^2}{2\sigma^2\cdot 2(1+\rho)}\right] \tag{29}$$

This equation was compared to the Gauss function. It is obvious from the comparation that the sum of two gaussian distributed variables with expectation 0, variance σ^2 and correlation coefficient ρ is once again gaussian distributed with expectation 0 and variance $2\sigma^2(1+\rho)$.

III. Entropy of Noisy Signals

To find the entropy of the signal at the decision device, some simplifications are carried out in (12). The term $\frac{(x+\mathrm{E_D})}{\sqrt{2}\sigma_D}$ in the first integral is replaced by t; for the second integral, $t = \frac{x-\mathrm{E_D}}{\sqrt{2}\sigma_D}$ is used for the simplification. Due to this replacement, the new integrant will be $dt = \frac{dx}{\sqrt{2}\sigma_D}$. After applying this, the two integrals can be transformed to one. It is now possible to make all terms inside the integral only dependent on the quotient $\frac{\mathrm{E_D}}{\sigma_D}$. Just one term, which has to be added to the integral, will contain σ_D. Using the expression $g(t)$ as defined in (15),

$$\mathrm{H}(Y) = \log_2(2\sqrt{2\pi}\sigma_D) - \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-t^2}\cdot g(t)dt \tag{30}$$

The two terms in (13) both express the intropy of gaussion noise, wich is known to be

$$\mathrm{H(WGN)} = \log_2(\sqrt{2\pi}\sigma_D) \tag{31}$$

The fact that the mean of the distributions is non–zero does not change their entropy. Thus, it is obvious that

$$\mathrm{H}(Y|X) = \log_2(\sqrt{2\pi}\sigma_D) \tag{32}$$

The final result, which is already given in (14), is easily found by substracting (32) from (30).