On the Power Spectral Density of Time-Hopping Impulse Radio

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Abstract—**The power spectral density (PSD) of time-hopping (TH) Ultra Wide Band (UWB) signals plays a major role in key aspects like coexistence with conventional radio systems. In the past, several papers have been published, presenting on the one hand the effects of modulation and timing jitter on the PSD assuming random TH codes. On the other hand, papers have been presented dealing with the PSD of specific TH codes without modulation. This paper presents a mathematical frame work that enables the evaluation of the PSD of a modulated impulse radio signal using a deterministic TH code. Besides being of theoretical interest, our results can be the starting point for the development of TH code design criteria aimed at the spectral shaping of the UWB signal.**

Index terms— **PSD, UWB, Impulse Radio, Time-hopping**

1. INTRODUCTION

Almost all communications systems in use today employ a sinusoid as an elementary waveform, on which information is mapped via some sort of modulation. The result is that signal energy is concentrated in a well defined frequency band, which makes noise and interference suppression relatively easy, e.g. by means of band-pass filtering. Unfortunately, such narrow band systems are inherently sensitive to fading. To obtain a robust wireless communication system, a fading margin has to be respected resulting in a lower capacity. Furthermore, spectral resources are divided into many narrow frequency bands causing the spectral resources to be fragmented.

In the last ten years the interest in ultra wide band (UWB) technology has grown [1], [2] and [3]. Not only due to its promise to re-use rare spectrum, but also due to its inherent resilience against fading leading to increased capacity in multipath environments. Additionally, generation of UWB signals requires low complexity, if ultra-short pulses are transmitted. A system deploying this technique is often referred to as an impulse radio (IR). Due to its high bandwidth a UWB signal is able to resolve its surrounding with high resolution, which in principle allows a single UWB device to be used for communication and radar applications.

Currently, regulation authorities in both Europe and the US are in the process of developing legislation for UWB signals. Clearly, no gigahertz bandwidth at the lower frequencies will be allocated for UWB exclusively. The US telecommunications regulator (FCC) has indicated that UWB devices most probably have to operate within limits described in Part 15 of FCC regulation. These limitations set boundaries on the transmit signal in both the frequency domain (limited power in a 10 MHz bandwidth) and the time domain (limited peak to mean, depending on the signal bandwidth). European regulatory bodies will most likely set similar limitations on the transmit signal.

Although UWB signals are alike in the frequency domain, they are diverse in the time domain. An important type of UWB signals (e.g.[3]) is constituted by a sequence of very short pulses, which occur pseudo-randomly in time. This pseudo-randomness is generated by a time hopping (TH) code. These signals can be modulated in several ways, including pulse amplitude modulation (PAM) and pulse position modulation (PPM). For this type of UWB signal, the TH code and the modulation scheme shall be designed such that reliability and throughput of the UWB system are maximized, without violating regulation. Furthermore, interference with narrowband systems should be kept to a minimum to accelerate the acceptance of UWB technology. A good understanding of the power spectral density (PSD) of UWB signals and how it is influenced by both the TH code and the modulation is mandatory to achieve these goals.

In the past, papers have been published on the PSD of UWB signals. In [4], the PSD of a modulated TH UWB signal is computed assuming a random TH code. However, the role of the TH code is not explicitly identified. In [5], the PSD of an IR employing a finite TH code is investigated without addressing the effect of modulation. In this paper we will derive the PSD of a modulated TH UWB signal in a form that allows to explicitly study the effects of the TH code and of the modulation on the PSD shape. Besides being of theoretical interest, our results are the starting point for the development of code design criteria aimed at the spectral shaping of the UWB signal.

2. SIGNAL DEFINITION

In this section a format for the transmitted signal is introduced. A similar format is presented in [3], but some modifications are required. Namely, in an IR the ultra-short pulses are randomized by a pseudo-random TH code, which inevitable will repeat itself. Therefore, it is convenient to describe firstly the waveform $s_p(t)$ transmitted in a single repetition period T_{TH}.

In order to construct the waveform, the total period T_{TH} contains N_b equally sized time intervals of length T_b, which is the symbol duration. Note that N_bT_b may be smaller as T_{TH}. On its turn, the symbol duration T_b contains N_s equally sized time intervals named frame T_f, again T_fN_s may be smaller as T_f. To form a waveform, a pulse is allocated inside each frame. Its position is dictated by the TH code. Specifically, each frame is divided into N_h chips of duration T_c. Logically, the TH code is a sequence of elements with an integer value between 0 and N_h-1. Again N_hT_c is only upper bounded by T_f. As a result, the waveform is a train containing N_bN_s elementary pulses. Here the elementary pulse is the so-called impulse function or delta function. Any other pulse shape is obtained by filtering the output of the process. Due to the waveform structure, it is convenient to view the TH code as a sequence of N_b sub-sequences or better sub-TH codes each containing N_s elements. We will denote by $c_{l,h}$ the h-th element of the l-th sub-TH code. To complete the waveform description, we have to consider how information symbols are modulated onto this waveform. We will consider both PAM and PPM, thus it is convenient to denote an information symbol as a pair <*a*,*b*>, where *a* belongs to a given PAM constellation while *b* specifies the multitude of elementary time shift T_Δ and is thus an integer number. During a TH code period, N_b information symbols are transmitted. Each combination of information symbols will be is denoted by an identifier *p*. The amplitude and time position of the *l*-th symbol of the *p*-th waveform will be denoted by a_l^p and b_l^p, and are assumed to be i.i.d. in respect to *l* and independent of each other. Based on the above description the transmitted waveform during a TH-period can be written as

$$s_p(t) = \sum_{l=0}^{N_b-1}\sum_{h=0}^{N_s-1} a_l^p \delta(t - lT_b - b_l^p T_\Delta - c_{l,h}T_c - hT_f) \qquad (1)$$

The waveform of Eq.1 carriers N_b information symbols, which are embedded in *p*. In order to transmit an infinite amount of information symbols, a sequence ***p*** of RVs is considered. The transmit signal $y(t)$ of an IR system is obtained by,

$$y(t) = \sum_k s_{p_k}(t - kT_{TH} - \Theta) \qquad (2)$$

where p_k denotes the k-th block of N_s information symbols and Θ is a random variable (RV), independent from the selected waveform and uniformly distributed along the interval $[0,\ldots,T_{TH}\rangle$, making the data-signal $y(t)$ a correlation-stationary process.

3. COMPUTATION OF THE PSD

In the appendix the PSD for the signal $y(t)$ is derived as,

$$P_y(\omega) = \frac{1}{T_{TH}} E\left\{ \left|S_p(\omega)\right|^2 \right\} + \frac{1}{T_{TH}} E\left\{ S_p(\omega) S_q^*(\omega) \right\} \left[\Pi_{T_{TH}}(\omega) - 1 \right], \qquad (3)$$

where *p* and *q* are two independent random variables with the same probability distribution as p_k. Furthermore, $S_p(\omega)$ is the Fourier Transform (FT) of $s_p(t)$,

$$S_p(\omega) = \sum_l a_l^p e^{-j\omega b_l^p T_\Delta} T_l(\omega), \qquad (4)$$

where $T_l(\omega)$ is the FT of the *l*-th sub-TH code given by,

$$T_l(\omega) = \sum_h e^{-j\omega c_{l,h}T_c} e^{-j\omega hT_f} e^{-j\omega lT_b} \qquad (5)$$

Furthermore, $\Pi_T(\omega)$ is the FT of an impulse train with period *T*.

$$\Pi_T(\omega) = \frac{1}{T}\sum_k \delta(\omega - \frac{2\pi k}{T}) \qquad (6)$$

Let us concentrate on the first expectation of Eq. 3. Taking the expectation into the summation and using the fact that a_l^p and b_l^p are independent, we obtain

$$E\left\{ \left|S_p(\omega)\right|^2 \right\} = \sum_{l,n} E\left\{a_l^p a_n^p\right\} E\left\{ e^{-j\omega(b_l^p - b_n^p)T_\Delta} \right\} T_l(\omega) T_n^*(\omega) \qquad (7)$$

The expectations in Eq.7 can take only two values, depending on whether $l = n$ or $l \neq n$. Separation of both cases gives

$$\begin{aligned} E\left\{ \left|S_p(\omega)\right|^2 \right\} &= \sum_{l,n=l} E\left\{a_l^p a_n^p\right\} E\left\{ e^{-j\omega(b_l^p - b_n^p)T_\Delta} \right\} T_l(\omega) T_n^*(\omega) \\ &\sum_{l,n\neq l} E\left\{a_l^p a_n^p\right\} E\left\{ e^{-j\omega(b_l^p - b_n^p)T_\Delta} \right\} T_l(\omega) T_n^*(\omega) \end{aligned} \qquad (8)$$

To clearly distinguish both cases, new variables are introduced

$$\begin{aligned} R_0^a &= E\{a_l^p a_n^p\} && \text{if } l = n, \\ R_1^a &= E\{a_l^p a_n^p\} && \text{if } l \neq n, \\ R_0^b(\omega) &= 1 && \text{if } l = n, \\ R_1^b(\omega) &= E\left\{ e^{-j\omega(b_l^p - b_n^p)T_\Delta} \right\} && \text{if } l \neq n. \end{aligned} \qquad (9)$$

Substitution of these variables into Eq.8 and re-ordering gives,

$$\begin{aligned} E\left\{ \left|S_p(\omega)\right|^2 \right\} &= \{R_0^a - R_1^a R_1^b(\omega)\} \sum_l T_l(\omega) T_n^*(\omega) \\ &+ R_1^a R_1^b(\omega) \sum_{l,n} T_l(\omega) T_n^*(\omega). \end{aligned} \qquad (10)$$

Having solved the first expectation in Eq.3, let us concentrate on the second expectation of Eq.3,

$$E\left\{ S_p(\omega) S_q^*(\omega) \right\} = \sum_{l,n} E\left\{a_l^p a_n^q\right\} E\left\{ e^{-j\omega(b_l^p - b_n^q)T_\Delta} \right\} T_l(\omega) T_n^*(\omega) \qquad (11)$$

The waveforms $s_p(t)$ and $s_q(t)$ are generated by two i.i.d. processes. Therefore, the expectations in Eq.11 are independent of *l* and *n* and equal to the case $l \neq n$ of Eq.9. Therefore,

$$E\left\{ S_p(\omega) S_q^*(\omega) \right\} = R_1^a R_1^b(\omega) \sum_{l,n} T_l(\omega) T_n^*(\omega). \qquad (12)$$

Combining Eqs.10 and 12 with Eq.3 and some re-ordering gives the following general mathematical expression for the PSD,

$$\begin{aligned} P_y(\omega) &= \frac{1}{T_{TH}} \left\{ \{R_0^a - R_1^a R_1^b(\omega)\} \sum_l T_l(\omega) T_l^*(\omega) \right\} + \\ &\frac{1}{T_{TH}} \left\{ R_1^a R_1^b(\omega) \sum_{l,n} T_l(\omega) T_n^*(\omega) \right\} \Pi_{T_{TH}}(\omega). \end{aligned} \qquad (13)$$

4. THE INFLUENCE OF MODULATION

As in [1], we see that the PSD of a TH impulse radio (See Eq.13) consists of a continuous and discrete PSD component. How the total power is distributed over both components depends partly on the expected values of Eq.9. The "spikiness" of selected parts of the PSD can be reduced by altering the modulation parameters. Furthermore, the continuous PSD component depends only on the auto-correlation spectrum of the sub-TH codes. Let us continue with a more specific analysis of the PSD for some concrete cases.

4.1 Limiting Case: No Modulation

Before investigating the effect of modulation, let us focus on the case of no modulation, since this result is well known, e.g. [5]. The expectations of Eq.9 become

$$R_0^a = R_1^a = A^2 \;, \qquad R_0^b(\omega) = R_1^b(\omega) = 1 \;, \tag{14}$$

where A represents the amplitude of an impulse. As a result,

$$P_y(\omega) = \frac{A^2}{T_{TH}} \left\{ \sum_{l,n} T_l(\omega) T_n^*(\omega) \right\} \left[\Pi_{T_{TH}}(\omega) \right], \tag{15}$$

which is in agreement with literature.

4.2 2-PPM modulated TH impulse radio

A modulation technique often deployed in a IR is binary PPM modulation. Since all waveforms are equi-probable, the bits become independent with equi-probable values. In this case, the expectations of Eq.9 become,

$$R_0^a = R_1^a = A^2 \;, \quad R_1^b(\omega) = \left(1 + \cos(\omega T_\Delta)\right)/2 \;, \tag{16}$$

which combined with Eq.13 gives

$$\begin{aligned} P_y(\omega) = {} & \frac{A^2}{T_{TH}} \left\{ \{ \frac{1 - \cos(\omega T_\Delta)}{2} \} \sum_{l} T_l(\omega) T_l^*(\omega) \right\} \\ & + \frac{A^2}{T_{TH}} \left\{ \frac{1 + \cos(\omega T_\Delta)}{2} \sum_{l,n} T_l(\omega) T_n^*(\omega) \right\} \left[\Pi_{T_{TH}}(\omega) \right] \end{aligned} \tag{17}$$

In Fig. 1 the PSD is compared to simulation results. For illustrative reasons only part of the spectrum is depicted.

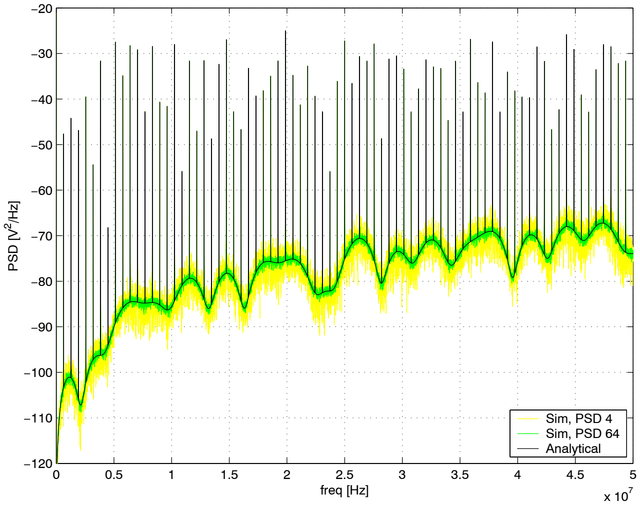

Fig. 1 PSD of a PPM TH UWB signal.

The deployed variable of section 2 are T_c =10ns, N_h=12, N_b=4, N_s= N_h/N_b, T_f=N_hT_c, T_b=N_sT_f, T_{th}=N_bT_b, T_Δ=1ns. In total three pulses are transmitted per symbol. The TH-code is constructed according to [7]. The prime used is equal to N_h+1. For this construction technique, the TH code length is equal to N_h, with the following content {0,6,8,9,7,10,1,4,2,3,5,11}. Note that the code construction technique is chosen arbitrary and not based on its spectral properties. The simulation produced several uncorrelated time intervals of 199681ns with a sample period of 1ns. The resulting PSD has been averaged over N intervals (denoted in the fig. by PSD N). To enable comparison, the continuous part of the analytical PSD is adjusted to the resolution bandwidth. The simulated PSD clearly converges to the derived PSD.

4.3 2-PAM and 2-PPM modulated TH impulse radio

The last modulation scheme that we considered, deploys a combination of antipodal binary-PAM and binary-PPM. If the sign and time-position are generated by a i.i.d. process, the expectations of Eq.9 are,

$$R_0^a = A^2 \;, \quad R_1^a = 0 \;, \quad R_1^b(\omega) = \left(1 + \cos(\omega T_\Delta)\right)/2 \;. \tag{18}$$

with the following corresponding PSD.

$$P_y(\omega) = \frac{A^2}{T_{TH}} \left\{ \sum_{l} T_l(\omega) T_l^*(\omega) \right\} \tag{19}$$

All variables remained the same as in paragraph 4.2, with the exception that the sign of the symbol is also modulated. Fig. 2 shows both the analytical and simulated PSD, which are again in agreement. Furthermore, the PSD is a continuous function and independent of the pulse position modulation. These effects are always observed if the expectation for the amplitude is zero.

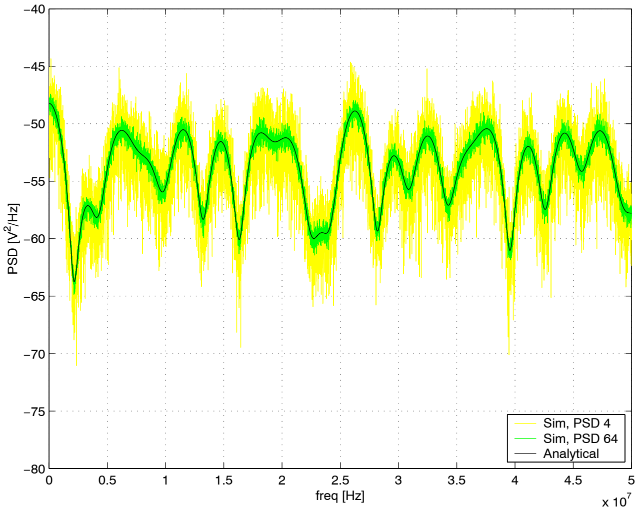

Fig. 2 PSD of a PAM and PPM TH UWB signal.

5. CONCLUSIONS

Closed form expressions for the PSD UWB signals deploying deterministic TH codes and employing different types of modulation are derived based on stochastic signal theory. The validity of the expressions is proven by means of comparison with simulation results and a limiting case. The results can be used as starting point for the development of code design criteria aimed at the spectral shaping of the UWB signal Note that the derived expressions for the PSD allow for the evaluation of both analogue and digital modulation techniques. Furthermore, the obtained expressions can also include other i.i.d. stochastic processes like timing jitter.

6. APPENDIX

In this appendix we show that the spectrum of the signal $y(t)$ of Eq.2 is given by Eq.3. To this end let us first recall and generalize the format of the waveform of Eq.2. Consider an i.i.d. sequence $\boldsymbol{d}$ where each element d_k is a RV with a given distribution. Suppose that for any possible value of d_k there exists a real Fourier transformable waveform denoted by $s_{d_k}(t)$. Form the waveform

$$y(t) = \sum_k s_{d_k}(t - kT - \Theta) \tag{20}$$

where T refers to the symbol time and Θ is a RV independent from the d_k and uniformly distributed from 0 until T. The signal of Eq.20 has the same format as the signal of Eq.2. In [6] it was shown that this is a correlation-stationary process and its spectrum was computed. In order to compute the spectrum we have to evaluate the autocorrelation of $y(t)$ namely

$$\begin{aligned} &E\{y(t)y(t+\tau)\} \\ &= E\left\{\sum_{k,h} s_{d_k}(t - kT - \Theta) s_{d_h}(t + \tau - hT - \Theta)\right\} \end{aligned} \tag{21}$$

Bringing the expectation inside the sum and taking the expectation over Θ yields

$$= \sum_{k,h} E\left\{\frac{1}{T}\int_0^T s_{d_k}(t - kT - \Theta) s_{d_h}(t + \tau - hT - \Theta) d\Theta\right\} \tag{22}$$

the expectation now involves the sequence d_k only. More specifically the expectation is on the two independent RV d_k and d_h unless $h = k$ in which case a single RV d_k is involved. For these reasons it is convenient to break the summation into two parts one for $h = k$ and the other in $h \neq k$. In addition the expectation is the same for any pair of i.i.d. RV d_k, d_h. It is convenient to introduce two RV say p and q independent and distributed like d_k, which will function as placeholders for d_k and d_h in the expectation. In summary we can write

$$\begin{aligned} &\sum_{k=h} E\left\{\frac{1}{T}\int_0^T s_p(t - kT - \Theta) s_p(t + \tau - hT - \Theta) d\Theta\right\} \\ &+ \sum_{k\neq h} E\left\{\frac{1}{T}\int_0^T s_p(t - kT - \Theta) s_q(t + \tau - hT - \Theta) d\Theta\right\} \end{aligned} \tag{23}$$

The autocorrelation of $y(t)$ is thus the sum of two terms. Let us study them separately. The first term can be written as,

$$\begin{aligned} &\sum_{k=h} E\left\{\frac{1}{T}\int_0^T s_p(t - kT - \Theta) s_p(t + \tau - hT - \Theta) d\Theta\right\} \\ &= E\left\{\frac{1}{T}\sum_k \int_0^T s_p(t - kT - \Theta) s_p(t + \tau - hT - \Theta) d\Theta\right\} \\ &= \frac{1}{T} E\left\{\int_{-\infty}^{\infty} s_p(t - \Theta) s_p(t + \tau - \Theta) d\Theta\right\} = \frac{1}{T} E\left\{R_s^p(\tau)\right\} \end{aligned} \tag{24}$$

Where $R_s^p(\tau)$ is the deterministic autocorrelation function of the waveform s_p. Let us now consider the sum for $h \neq k$. By substitution of h by $l = h - k$, we obtain

$$\begin{aligned} &\sum_{k\neq h} E\left\{\frac{1}{T}\int_0^T s_p(t - kT - \Theta) s_q(t + \tau - hT - \Theta) d\Theta\right\} \\ &= \sum_k \sum_{l\neq 0} E\left\{\frac{1}{T}\int_0^T s_p(t - kT - \Theta) s_q(t + \tau - (k+l)T - \Theta) d\Theta\right\} \\ &= \sum_{l\neq 0} E\left\{\frac{1}{T}\sum_k \int_0^T s_p(t - kT - \Theta) s_q(t + \tau - (k+l)T - \Theta) d\Theta\right\} \\ &= \frac{1}{T}\sum_{l\neq 0} E\left\{\int_{-\infty}^{\infty} s_p(t - \Theta) s_q(t + \tau - lT - \Theta) d\Theta\right\} \\ &= \frac{1}{T}\sum_{l\neq 0} E\left\{R_s^{q,p}(\tau - lT)\right\} \end{aligned} \tag{25}$$

where $R_s^{q,p}(\tau)$ is the deterministic cross correlation between the two waveforms $s_p(t)$ and $s_q(t)$. By replacing Eq.23 and Eq.24 in Eq.22, we obtain

$$E\{y(t)y(t+\tau)\} = \frac{1}{T} E\left\{R_s^p(\tau)\right\} + \frac{1}{T}\sum_{l\neq 0} E\left\{R_s^{q,p}(\tau - lT)\right\} \tag{26}$$

The expectations for both the auto- and cross-correlation depends on τ only. We can thus take the FT over τ to obtain the spectrum. After some algebra the spectrum can be written as in Eq.3, i.e. in the following form

$$P_y(\omega) = \frac{1}{T} E\left\{ \left|S_p(\omega)\right|^2 \right\} + \frac{1}{T} E\left\{S_p(\omega) S_q^*(\omega)\right\}\left[\Pi_T(\omega) - 1\right] \tag{27}$$

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